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Band Invariants and Closed Trajectories on S^n

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1. INTRODUCTION

Let M be a compact Riemannian manifold, and let Δ denote its Laplace–Beltrami operator. The observed energy of a quantum particle whose configuration space is M and which is subject to the influence of a potential, $q \in C^\infty(M)$, can only take a discrete set of values, namely, the points in the spectrum of the Schrödinger operator $\Delta + q$. On the basis of physical considerations, we should expect that, for very high values of the energy, the particle behaves like a classical particle. This general "correspondence principle" has been the source of many beautiful mathematical results, like the celebrated theorem of Weyl and Hörmander [H] and the results of Duistermaat and Guillemin on the spectrum of elliptic operators and periodic bicharacteristics [DG]. In this paper we look at the special case when M is the standard n-sphere. The spectrum of $\Delta + q$ forms "bands" of fixed width around the spectrum of the Laplacian, Δ. We show that, under certain assumptions on q, there is a correspondence between clustering in the bands and the existence of closed classical orbits, i.e., integral curves of the Hamiltonian $|\cdot|^2 + q$ on T^*S^n. We do this in two steps. First, in Sections 2–3, we prove a theorem on the existence of closed orbits. The proof is based on a result of Moser [M], which in turn is based on the "averaging method." We present a version of Moser's result adapted to our purposes (Theorem 2.4); then, by rescaling, we prove the existence of families of high-energy closed orbits, $\{\gamma_r\}$, on T^*S^n (Theorems 3.1 and 3.2). For example, in the case of the spherical pendulum ($n = 2$), the closed orbits predicted by our theorem are Huygens' horizontal trajectories (parallels below the equator).

In the second part of the paper we relate the asymptotic behavior of the periods $T(r)$ of the trajectories $\{\gamma_r\}$ to the spectrum of $\Delta + q$. It turns out that the eigenvalues in the kth band accumulate, as $k \uparrow \infty$, around the coefficient of the second term in the Taylor series of $T(r)$ at infinity. In dimension 2 we make this more precise by considering certain quasimodes

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associated with the family $\{\gamma_r\}$. Finally, Section 6 contains some formal computations pertaining to the averaging method which are necessary to handle the case of odd potentials.

2. Closed Orbits and the Averaging Method

Let (X, Ω) be a conic symplectic manifold. Thus, Ω is a symplectic form on the manifold X and we are given a free $\mathbb{R}^+$-action on X such that, for every $r \in \mathbb{R}^+$, $r^*\Omega = r\Omega$. Suppose that we are also given a homogeneous function of degree one,

$$H_0 : X \to \mathbb{R}^+, \tag{2.1}$$

which is a submersion with compact level surfaces. Furthermore, assume that the trajectories of Ξ_0, the Hamiltonian vector field associated with H_0, are all closed with the same period, say, 2π. The example we have in mind is $X = T^*S^n - \{0\}$ and $H_0 = |\cdot|$, the Riemannian norm function. In this section we will study what happens to the previous picture when we perturb H_0 a little.

Let $Z = H_0^{-1}(1)$, and introduce the quotient manifolds

$$B = X/S^1, \qquad \mathcal{O} = Z/S^1,$$

where the S^1-action on X (and on Z) is the one defined by the flow of Ξ_0. We will denote by $\pi : Z \to \mathcal{O}$ the natural projection. As is well known, the manifold $\mathcal{O}$ admits a unique symplectic form, ω, such that $\pi^*\omega = \Omega|_Z$. As for B, it inherits an $\mathbb{R}^+$-action induced by the given one on X. The function H_0 defines diffeomorphisms,

$$X \cong Z \times \mathbb{R}^+, \qquad B \cong \mathcal{O} \times \mathbb{R}^+.$$

Let $\{\ ,\ \}$ denote the Poisson bracket on X. Given a function, $G \in C^\infty(X)$, such that $\{G, H_0\} = 0$, there is a unique function $g \in C^\infty(B)$, whose pull-back to X is G. Let $\tilde{g} \in C^\infty(\mathcal{O})$ be the restriction of g to $\mathcal{O}$.

2.1. Proposition. *Suppose G is homogeneous of degree m and $\{G, H_0\} = 0$. Let $H_\varepsilon = H_0 + \varepsilon G$, and suppose that $\alpha \in \mathcal{O}$ is a critical point of $\tilde{g}$. Then, for small ε, $\gamma = \pi^{-1}(\alpha)$ is a periodic trajectory of H_ε which is non-degenerate if and only if α is a non-degenerate critical point of $\tilde{g}$. Furthermore, the period of γ as an orbit of H_ε is*

$$T_\varepsilon = \frac{2\pi}{1 + \varepsilon m g(\alpha)}. \tag{2.2}$$

Proof. We will prove that, if Θ is the Hamiltonian vector field associated with G, then

$$\Theta = mg(\alpha)\, \Xi_0 \tag{2.3}$$

at every point of γ. Let $h \in C^\infty(B)$ be the function whose pull-back to X is H_0. Since α is a critical point of $\tilde{g}$, by Lagrange multipliers there exists $a \in \mathbb{R}$ such that

$$(dg)_\alpha = a(dh)_\alpha. \tag{2.4}$$

Evaluating (2.4) on the infinitesimal generator of the $\mathbb{R}^+$-action on B and using the homogeneity assumptions, we find that $mg(\alpha) = a$. This implies (2.3). It follows easily that γ is a periodic orbit of H_ε with period given by (2.2).

Let ξ be the Hamiltonian vector field on $\mathcal{O}$ associated with $\tilde{g}$. The Poincaré map of the flow of H_ε restricted to Z associated with γ can be identified with

$$d(\exp \varepsilon T_\varepsilon \xi)\colon T_\alpha \mathcal{O} \to T_\alpha \mathcal{O}. \tag{2.5}$$

Let A be the infinitesimal generator of the one-parameter group of symplectomorphisms of $T_\alpha \mathcal{O}$, $t \mapsto d(\exp t\xi)$. It is well known that, if $(d^2\tilde{g})_\alpha$ is the Hessian of $\tilde{g}$ at α, then, for all $v, w \in T_\alpha$,

$$(d^2\tilde{g})_\alpha(v, w) = \omega_\alpha(Av, w).$$

Hence $(d^2\tilde{g})_\alpha$ is non-degenerate if and only if A does not have zero as an eigenvalue, that is, iff the Poincaré map (2.5) does not have one as an eigenvalue for small ε. Q.E.D.

Now suppose that we have a smooth, one-parameter family of functions $H_\varepsilon \in C^\infty(X)$ such that $H_{\varepsilon|\varepsilon=0}$ is the function (2.1). Could we prove a result similar to Proposition 2.1 for a general perturbation like this? The answer is yes; the following "averaging lemma" says that all perturbations are more or less equivalent, for small ε, to the ones considered previously.

2.2. Lemma. *For every integer $k \geqslant 1$ there exists a smooth family of canonical transformations, $\Psi_\varepsilon\colon X \to X$, such that $\Psi_0 = I$ and if $\Psi_\varepsilon^* H_\varepsilon = H_0 + \varepsilon H_1^\# + \cdots + (1/k!)\, \varepsilon^k H_k^\# + O(\varepsilon^{k+1})$ as $\varepsilon \to 0$, then $\{H_0, H_j^\#\} = 0$ for all j, $1 \leqslant j \leqslant k$.*

The proof is an application of the averaging method; see Section 6. In fact, using the formula (6.1), we can write down $H_1^\#, H_2^\#$. For every $H \in C^\infty(X)$, define

$$H^{\mathrm{av}} = \frac{1}{2\pi} \int_0^{2\pi} (\exp t\Xi_0)^*\, H\, dt \tag{2.6}$$

and

$$\tau(H) = \frac{-1}{2\pi} \int_0^{2\pi} dt \int_0^t (\exp s\Xi_0)^* \, H \, ds. \tag{2.7}$$

It is easy to see that $\{H_0, H^{\mathrm{av}}\} = 0$.

2.3. LEMMA. *Take $k \geqslant 2$ in Lemma 2.2 and let $H_1, H_2 \in C^\infty(X)$ be such that*

$$H_\varepsilon = H_0 + \varepsilon H_1 + \tfrac{1}{2}\varepsilon^2 H_2 + O(\varepsilon^2)$$

as $\varepsilon \to 0$. Then, with the notation of Lemma 2.2,

$$H_1^{\#} = H_1^{\mathrm{av}}$$

and $H_2^{\#} = [\{\tau(H_1), H_1\} + H_2]^{\mathrm{av}}$.

We can now state the main result of this section.

2.4. THEOREM (See Moser [M]). *Keeping the notation introduced above, let $h_1 \in C^\infty(\mathcal{O})$ be such that $\pi^* h_1 = H_1^{\mathrm{av}}|_Z$. Suppose that $\alpha \in \mathcal{O}$ is a non-degenerate critical point of h_1. Then there exists a family, $\{\gamma_\varepsilon\}$, of closed trajectories of H_ε defined for small ε and such that:*

(a) $\gamma_\varepsilon \subset \{H_\varepsilon = 1\}$,

(b) *γ_ε approaches $\pi^{-1}(\alpha)$ as $\varepsilon \to 0$.*

Furthermore, if H_1 is homogeneous of degree m, the period, T_ε, of γ_ε satisfies

$$T_\varepsilon = 2\pi[1 - \varepsilon m h_1(\alpha)] + O(\varepsilon^2), \tag{2.8}$$

as $\varepsilon \to 0$.

If $H_1^{\mathrm{av}} = 0$, then the above holds with h_1 replaced by the function h_2 such that

$$\pi^* h_2 = [\{\tau(H_1), H_1\} + \tfrac{1}{2} H_2]^{\mathrm{av}}|_Z$$

and with (2.8) replaced by

$$T_\varepsilon = 2\pi[1 - \tfrac{1}{2}\varepsilon^2 m h_2(\alpha)] + O(\varepsilon^3).$$

Proof. By Lemma 2.3, it is enough to prove the theorem with H_ε replaced by $H_\varepsilon^{\#}$. Let $U = H_0^{-1}(1-\delta, 1+\delta)$ for some δ, $0 < \delta < 1$. U is a tubular neighborhood of Z; let $p\colon U \to Z$ be its projection. By compactness of Z, the level surface,

$$Z_\varepsilon = (H_\varepsilon^{\#})^{-1}(1),$$

is contained in U as a section of p if ε is small enough. Let s_ε be such a section; it induces a diffeomorphism

$$s_\varepsilon\colon Z \to Z_\varepsilon.$$

Denote by Θ_ε the vector field on Z which is related via s_ε with the restriction to Z_ε of the Hamiltonian vector field of $H_\varepsilon^\#$. It is clearly enough to find closed trajectories of Θ_ε, for small ε, branching off from $\gamma_0 = \pi^{-1}(\alpha)$. This follows from Proposition 2.1; Θ_ε is a perturbation of order $O(\varepsilon^2)$ of the Hamiltonian vector field associated with $H_0 + \varepsilon H_1^{\mathrm{av}}$. Non-degeneracy of α implies the non-degeneracy of γ_0 as a trajectory of $H_0 + \varepsilon H_1^{\mathrm{av}}$, so it survives small perturbations. The statement about T_ε follows from (2.2).

Q.E.D.

3. A Classical Particle on S^n

Consider $X = T^*S^n - \{0\}$, $H_0 = |\cdot|$ the Riemannian norm on X, and a potential function, $q \in C^\infty(S^n)$, which we think of as a function on X. The motion of a classical particle on S^n subject to the potential q is then described by the flow on X of the Hamiltonian vector field associated with $H_0^2 + q$.

Keeping the notation of Section 2, let $Z = H_0^{-1}(1)$ be the unit cosphere bundle of S^n. Geodesic flow defines an S^1-action on Z; $\mathcal{O} = Z/S^1$ is the space of oriented geodesics. Let $\pi\colon Z \to \mathcal{O}$ be the natural projection and $\rho\colon Z \to S^n$ the cotangent fibration. We will denote by $\tilde{q} \in C^\infty(\mathcal{O})$ the generalized Radon transform of q, given by

$$\tilde{q} = \frac{1}{2\pi}\,\pi_*\rho^*(q).$$

3.1. Theorem. *Let $\alpha \in \mathcal{O}$ be a non-degenerate critical point of $\tilde{q}$. Then there exists a smooth family of closed curves in X, $\{\gamma_r\}$, defined for r large, and such that*

(a) *γ_r is an integral curve of $H_0^2 + q$ on the energy surface $\{H_0^2 + q = r^2\}$,*

(b) *$r^{-1}(\gamma_r)$ approaches $\gamma = \pi^{-1}(\alpha)$ as $r \uparrow \infty$.*

Furthermore, if $T(r)$ denotes the period of γ_r, then

$$\frac{r}{\pi}\,T(r) = 1 + \frac{1}{2}\,\tilde{q}(\alpha)\,r^{-2} + O(r^{-4}) \tag{3.1}$$

as $r \uparrow \infty$.

Proof. Introduce the family of functions

$$H_\varepsilon = (H_0^2 + \varepsilon q)^{1/2}.$$

Then, as $\varepsilon \to 0$, $H_\varepsilon = H_0 + \varepsilon H_1 + \frac{1}{2}\varepsilon^2 H_2 + O(\varepsilon^3)$ where $H_1 = \frac{1}{2} q H_0^{-1}$ and $H_2 = -\frac{1}{4} q^2 H_0^{-3}$. We can now apply Theorem 2.4 to obtain the existence of $\{\overline{\gamma_\varepsilon}\}$, a family of closed trajectories of H_ε, where $\overline{\gamma_\varepsilon} \subset \{H_\varepsilon = 1\}$. Moreover, the period function, $\bar{T}_\varepsilon$, of $\bar{\gamma}_\varepsilon$ satisfies $\bar{T}_\varepsilon = 2\pi[1 + \frac{1}{2}\tilde{q}(\alpha)\,\varepsilon] + O(\varepsilon^2)$.

Having found closed orbits for H_ε we will obtain closed orbits for $H = (H_0^2 + q)^{1/2}$ by rescaling.

Notice that, for every $r \in \mathbb{R}^+$,

$$r^* H = r H_{r^{-2}}. \tag{3.2}$$

For large r, define $\gamma_r = r(\bar{\gamma}_{r^{-2}})$. It is easy to see that $\gamma_r \subset \{H = r\}$. Moreover, since $r^*\Omega = r\Omega$ (where Ω is the symplectic form on X),

$$r_*(\Xi_{r^*H}) = r\Xi_H. \tag{3.3}$$

We have designated by Ξ_G the Hamiltonian vector field associated with the function G. Equations (3.2) and (3.3) imply

$$r_*(\Xi_{H_{r^{-2}}}) = \Xi_H.$$

Hence γ_r is an integral curve of H. Furthermore if $T^{\#}(r)$ denotes the period of γ_r as a trajectory of Ξ_H, then

$$T^{\#}(r) = \bar{T}(r^{-2}).$$

Since $\Xi_{H^2} = 2H\Xi_H$ and $\gamma_r \subset \{H = r\}$, $2rT(r) = T^{\#}(r)$. The theorem follows.
Q.E.D.

One case in which Theorem 3.1 does not give any information is when $\tilde{q}$ is identically zero. This happens if and only if q is an odd function; see [G1]. In that case we must consider the function

$$\frac{1}{4} H_0^{-3}(q^2)^{\mathrm{av}} - \frac{1}{8\pi} H_0^{-2} \int_0^{2\pi} dt \int_0^t \{(\exp t\Xi_0)^* q, (\exp s\Xi_0)^* q\}\, ds. \tag{3.4}$$

3.2. Theorem. *If q is an odd function,* (3.4) *Poisson-commutes with H_0. Moreover, if $\tilde{\tilde{q}} \in C^\infty(\mathcal{O})$ is the corresponding function on $\mathcal{O}$ and α is a non-degenerate critical point of $\tilde{\tilde{q}}$, then there exists a family $\{\gamma_r\}$ of closed orbits satisfying* (a), (b) *of Theorem* 3.1. *If $T(r)$ is the period of γ_r, then*

$$T(r) = \pi r^{-1} + \tfrac{3}{2}\pi\tilde{\tilde{q}}(\alpha)\, r^{-5} + O(r^{-7})$$

as $r \uparrow \infty$.

The proof is identical to the proof of Theorem 3.1. The function (3.4) is the function $H_2^{\#}$ of Lemma 2.3 when $q^{\mathrm{av}}=0$; see Lemma 6.3, particularly Eq. (6.2).

4. Band Asymptotics; Odd Potentials

Let $q \in C^{\infty}(S^n)$ be a real-valued function, and let Δ be the (positive) Laplace–Beltrami operator on S^n. The spectrum of the Schrödinger operator, $\Delta + q$, forms bands of bounded width around the spectrum of Δ. More precisely, if $\lambda_k = k(k+n-1)$ is the kth eigenvalue of Δ, there is a constant C such that

$$\operatorname{Spec}(\Delta + q) \subset \bigcup_{k=0}^{\infty} [\lambda_k - C, \lambda_k + C].$$

For large k, the intervals $I_k = [\lambda_k - C, \lambda_k + C]$ are disjoint; let $\mu_1^{(k)}, \ldots, \mu_{d_k}^{(k)}$ be the eigenvalues of $\Delta + q$ in I_k, counted with multiplicity. The distribution of the shifted eigenvalues $\lambda_j^{(k)} = \lambda_k - \mu_j^{(k)}$ has very interesting asymptotic properties; see [G2], [W], [C]. Specifically, let

$$\nu_k(\lambda) = \frac{1}{d_k} \sum_{j=1}^{d_k} \delta(\lambda - \lambda_j^{(k)})$$

be the measure describing the distribution of eigenvalues in the kth band. Then:

4.1. Theorem (Weinstein [W]). *As $k \uparrow \infty$, the measures ν_k converge weakly to the measure*

$$\tilde{q}_*(dv), \tag{4.1}$$

where dv is the multiple of the Liouville measure on $\mathcal{O}$ such that $\int_{\mathcal{O}} dv = 1$, and $\tilde{q}$ is as in Section 3.

Recall that $\tilde{q} = 0$ if and only if q is an odd function. We will prove:

4.2. Theorem. *If q is odd, the sequence of measures $k^2 \nu_k$ converges weakly to the measure*

$$\tilde{\tilde{q}}_*(dv), \tag{4.2}$$

where $\tilde{\tilde{q}}$ is the function of Theorem 3.2.

The remainder of this section is devoted to proving Theorem 4.2. We begin by recalling the following basic result.

4.3. LEMMA (Guillemin [G2]). *There exists a zeroth-order self-adjoint pseudodifferential operator* (ΨDO), Q, *such that*

(a) Q *commutes with* Δ,

(b) $\Delta + q$ *and* $\Delta + Q$ *are unitarily equivalent via an invertible* ΨDO.

The proof is an application of the averaging method (compare with Lemma 2.2). If V_k denotes the space of kth-order spherical harmonics, then $Q(V_k) \subset V_k$ and it is clear that the spectrum of $Q|_{V_k}$ is precisely $\{\lambda_1^{(k)}, ..., \lambda_{d_k}^{(k)}\}$; in particular $d_k = \dim V_k$. Thus, in order to prove Theorem 4.2 we need to get a hold of the operator Q.

Let A be the operator on $L^2(S^n)$ whose restriction to V_k is "multiplication by k." A is an elliptic, first-order, self-adjoint ΨDO whose principal symbol is $H_0 = |\cdot|$, the Riemannian norm function. For every ΨDO, P, let

$$P^{\text{av}} = \frac{1}{2\pi} \int_0^{2\pi} U(t)\, PU(t)\, dt \tag{4.3}$$

and

$$i\tau(P) = \frac{1}{2\pi i} \int_0^{2\pi} dt \int_0^t U(s)\, PU(-s)\, ds, \tag{4.4}$$

where $U(t) = \exp itA$. If p is the principal symbol of P, then, by Egorov's theorem, P^{av} and $\tau(P)$ are ΨDO of the same order as P and with symbols p^{av}, $i\tau(p)$, respectively (see Eqs. (2.6), (2.7)). For clarity, denote by M_q the operator "multiplication by q." We can then describe the operator Q, modulo operators of order (-3), as follows:

4.4. LEMMA. *We can choose the operator* Q *of Lemma* 4.3 *to be of the form*

$$Q = (M_q)^{\text{av}} + R,$$

where R *is a* ΨDO *of order* (-2) *whose principal symbol, restricted to* Z, *equals*

$$\frac{1}{4}\,[(q^2)^{\text{av}} - (q^{\text{av}})^2] + \frac{1}{2}\,\{\tau(q), q^{\text{av}}\}$$
$$- \frac{1}{8\pi} \int_0^{2\pi} dt \int_0^t \{(\exp t\Xi)^* q, (\exp s\Xi)^* q\}\, ds. \tag{4.5}$$

Proof. One can easily check that, if $B = A + (n-1)/2$, then $B^2 = \Delta +$

$(n-1)^2/4$. Hence, it is enough to prove the lemma for the operator B^2+M_q. Let E be defined by

$$(B^2+M_q)^{1/2}=B+E.$$

This equation is easily seen to be equivalent to

$$M_q=2BE+[E,B]+E^2.$$

This shows that E is of order (-1) with principal symbol $\frac{1}{2}H_0^{-1}q$. Hence, we can take the operator

$$E_1=\tfrac{1}{4}(B^{-1}M_q+M_qB^{-1}) \tag{4.6}$$

as an approximation to E to order (-1). Next, let $E_2=E-E_1$. As we will see below, E_2 is of order (-3). Hence we can apply Lemma 6.3 to $P=A+E_1+E_2$. It follows that, modulo operators of order (-4), $(B^2+M_q)^{1/2}$ is unitarily equivalent to

$$B+E_1^{\text{av}}+\tfrac{1}{2}[\tau(E_1),E_1]^{\text{av}}+E_2^{\text{av}}. \tag{4.7}$$

The choice of E_1 was made so that $2BE_1^{\text{av}}=M_q^{\text{av}}$. Hence, when we square (4.7) we obtain an operator of the form $M_q^{\text{av}}+R$, where R is of order (-2). The final ingredient needed to show that the principal symbol of R is (4.5) is a computation of the principal symbol of E_2:

4.5. Lemma. *The operator E_2 is of order (-3) and principal symbol*

$$-\tfrac{1}{8}H_0^{-3}q^2+\tfrac{1}{8}H_0^{-1}\{H_0,\{q,H_0^{-1}\}\}. \tag{4.8}$$

Proof. For every ΨDO, P, on S^n, let

$$\mathscr{L}(P)=2BP+[P,B].$$

By definition, E is the ΨDO satisfying

$$M_q=\mathscr{L}(E)+E^2.$$

Replacing E by E_1+E_2 and neglecting terms of order (-3), we get

$$M_q=\mathscr{L}(E_1)+\mathscr{L}(E_2)+E_1^2 \qquad \operatorname{mod}(-3).$$

It is not hard to see that

$$\mathscr{L}(E_1)=M_q+\tfrac{1}{4}[B,[M_q,B^{-1}]],$$

and so the previous equation becomes

$$2BE_2+[E_2,B]=-\tfrac{1}{4}[B,[M_q,B^{-1}]]-E_1^2,$$

modulo operators of order (-3). The right-hand side of this identity is of order (-2); hence E_2 must be of order (-3) and its symbol satisfies

$$2H_0\sigma(E_2) = -\tfrac{1}{4}\sigma([B, [M_q, B^{-1}]]) - \tfrac{1}{4}H_0^{-2}q^2.$$

This proves Lemma 4.5. The remaining computations in the proof of Lemma 4.4 will be omitted. (See the proof of Lemma 6.3.) Q.E.D.

With Lemma 4.4 at hand, the proof of Theorem 4.2 is easy. If q is odd, then $(M_q)^{\text{av}} = 0$ and conversely; see [G3]. By Proposition 4.4, the operator Q is of order (-2) and its principal symbol, considered as a function on $\mathcal{O}$, is precisely $\tilde{\tilde{q}}$ (compare (4.5) and (3.4) when $q^{\text{av}} = 0$). Theorem 4.2 now follows immediately from the theory of band invariants; see [W], [G2], [U]. The function $i\pi\tilde{\tilde{q}}$ is precisely the symbol of Guillemin's return operator,

$$W = (-1)^{n-1} \exp 2\pi i \left(\Delta + \frac{(n-1)^2}{4} + q\right)^{1/2} - I,$$

when q is odd (see [G3; Theorem 1]). This also follows directly from our previous computations.

5. Clustering within the Bands

Let $c \in \mathbb{R}$ be a singularity of the measure (4.1). Then c is necessarily a critical value of $\tilde{q}$. If $c = \tilde{q}(\alpha)$, where α is a non-degenerate singularity of $\tilde{q}$, then Theorem 3.1 holds and we find closed orbits, $\{\gamma_r\}$, for the classical Hamiltonian $H_0^2 + q$ whose periods are approximately $\pi r^{-1} + (\pi/2)\, c r^{-3}$, for r large. Thus the singularity c appears as the second term in the asymptotic expansion of the periods of the γ_r. A similar statement holds for odd potentials, with $\tilde{q}$ replaced by $\tilde{\tilde{q}}$.

On the other hand, if c is a singularity of the limit of the measures γ_k (Theorem 4.1), the spectrum of $\Delta + q$ must accumulate around the values $\lambda_k + c$, as $k \uparrow \infty$. Thus we have found another instance of the correspondence between clustering in the spectrum and existence of closed orbits. In this section we will attempt to make this more precise:

5.1. Theorem. *Let $c \in \operatorname{Image} \tilde{q}$, and let $\{\mu_k\}$ be a sequence of eigenvalues of $\Delta + q$ minimizing $|\lambda_k + c - \mu_k|$, for every k. Then*

$$|\lambda_k + c - \mu_k| = O(k^{-1/2}). \tag{5.1}$$

Furthermore, at least on S^2, if c is a critical value of $\tilde{q}$, (5.1) can be improved to $O(k^{-3/2})$.

Proof. Let $\alpha \in \mathcal{O}$ be such that $c = \tilde{q}(\alpha)$. Let $(x_1, x_2, ..., x_{n+1})$ be Cartesian coordinates in $\mathbb{R}^{n+1}$. Without loss of generality, we may assume that the geodesic corresponding to α is the equatorial geodesic, intersection of S^n with the (x_1, x_2)-plane. Let ϕ be the complex-valued function $x_1 + \sqrt{-1}\, x_2$ restricted to S^n. The L^2-closure of the set $\{\phi^k, k = 0, 1, 2, ...\}$ is the space of quasimodes associated with the equatorial geodesic ([G4], [U]). The proof of Theorem 5.1 consists in analyzing to what extent the functions $\{\phi^k\}$ are still good quasimodes for the operator $\Delta + Q$ of Lemma 4.3.

Let us introduce the following notation. For every ΨDO, P, on S^n, define

$$\sigma_P(k) = \frac{\langle P(\phi^k), \phi^k \rangle}{\langle \phi^k, \phi^k \rangle},$$

where $\langle\ ,\ \rangle$ designates the L^2-inner product on S^n. As Guillemin proves in [G4], the numbers $\sigma_P(k)$ possess an asymptotic expansion

$$\sigma_P(k) \sim \sum_{j=0}^{\infty} a_j k^{m-j} \tag{5.2}$$

as $k \uparrow \infty$, where m is the order of P and a_0 is the average of the principal symbol of P over the equatorial geodesic. We have elaborated on this in [U]; our results will enable us to get hold of the numbers

$$\Gamma(k) = \|\phi^k\|^{-1}\, \|(Q - \sigma(k))\, \phi^k\|$$

where $\sigma(k) = \sigma_Q(k)$ and $\|\cdot\|$ is the L^2-norm on S^n.

5.2. Lemma. *$\Gamma(k) = bk^{-1/2} + O(k^{-3/2})$ as $k \uparrow \infty$. Furthermore, in dimension 2,*

$$b = \frac{\sqrt{2}}{2}\, |\nabla \tilde{q}(\alpha)|,$$

and so $\Gamma(k) = O(k^{-3/2})$ if and only if α is a critical point of $\tilde{q}$.

Before proving this lemma, let us see how it implies Theorem 5.1. Let V_k be the space of kth-order spherical harmonics, and, for every k, let τ'_k be an eigenvalue of $(Q - \sigma(k)) \mid V_k$ with minimal absolute value. By the minimax principle,

$$(\tau'_k)^2 \leqslant \Gamma^2(k). \tag{5.3}$$

Let $\tau_k = \tau'_k + \sigma(k)$; this is an eigenvalue of Q. Inequality (5.3) is equivalent to

$$-\Gamma(k) \leqslant \tau_k - \sigma(k) \leqslant \Gamma(k),$$

which we shall write as

$$-\Gamma(k) \leqslant (\tau_k - c) + (c - \sigma(k)) \leqslant \Gamma(k). \tag{5.4}$$

By the remarks that we made about (5.2), we know that $c - \sigma(k) = O(k^{-1})$. This, together with (5.4) and Lemma 5.2, proves the theorem for the sequence of eigenvalues $\{\lambda_k + \tau_k\}$, and hence a fortiori for $\{\mu_k\}$.

Now a few words about the proof of Lemma 5.2. It is easy to see that

$$\Gamma^2(k) = \sigma_{Q^2}(k) - (\sigma(k))^2. \tag{5.5}$$

At this point we need a few results from [U] in order to get a hold of σ_{Q^2}.

5.3. LEMMA. *Let $\sigma(k) = c + c_1 k^{-1} + O(k^{-2})$. Then $\sigma_{Q^2}(k) = c^2 + (2cc_1 + D_1(\tilde{q}, \tilde{q})(\alpha))\, k^{-1} + O(k^{-2})$, where D_1 is a certain bidifferential operator on $\mathcal{O}$. In dimension two, $D_1(\tilde{q}, \tilde{q}) = \frac{1}{2} |\nabla \tilde{q}|^2$.*

The reader can find the proof of this lemma in Section 6 of [U]; see in particular Lemma 6.5. It follows that

$$\Gamma^2(k) = D_1(\tilde{q}, \tilde{q})(\alpha)\, k^{-1} + O(k^{-2}).$$

This proves Lemma 5.2. Q.E.D.

For odd potentials the corresponding theorem is:

5.4. THEOREM. *Let $q \in C^\infty(S^n)$ be odd and let $c \in \text{Image}\, \tilde{\tilde{q}}$. Then, if $\{\mu_k\}$ is as in Theorem 5.1,*

$$|\lambda_k + c - \mu_k| = O(k^{-3/2}).$$

Furthermore, in dimension two, the above estimate can be improved to $O(k^{-5/2})$ if c is a critical value of $\tilde{\tilde{q}}$.

6. FORMAL AVERAGING

In this section we present the averaging method in a formal setting. We want to be able to prove Lemmas 2.3 and 4.3 in a way as constructive as possible.

Let $\mathscr{g}$ be an (infinite-dimensional) Lie algebra, and let $\imath \geqslant 0$ be an integer. We will say that $\mathscr{g}$ is $\imath$-graded if we are given a graduation of $\mathscr{g}$ by subspaces, $\mathscr{g}_k \supset \mathscr{g}_{k-1}$, $\mathscr{g} = \bigcup_{k \in \mathbb{Z}} \mathscr{g}_k$ such that $[\mathscr{g}_k, \mathscr{g}_l] \subset \mathscr{g}_{k+l-\imath}$. For example, the algebra of ΨDO on a manifold is 1-graded if we let $\mathscr{g}_k$ consist of the ΨDO of order $\leqslant k$.

We will assume that we can exponentiate elements of $\mathscr{g}$ onto the adjoint group of $\mathscr{g}$ in such a way that the usual identity

$$(\exp P) \cdot Q = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{ad}_P^n(Q), \qquad P, Q \in \mathscr{g},$$

holds. As usual, $\operatorname{ad}_P(Q) = [P, Q]$ and we have denoted the action of the adjoint group of $\mathscr{g}$ on $\mathscr{g}$ by a dot. Since we are only interested in formal computations, we will not be concerned with questions of convergence.

Suppose that we are given a distinguished element, $A \in \mathscr{g}_\iota$, such that $\exp tA$ is periodic in t of period 2π. For every $Q \in \mathscr{g}$, let

$$Q^{\text{av}} = \frac{1}{2\pi} \int_0^{2\pi} (\exp tA) \cdot Q \, dt$$

and

$$\tau(Q) = \frac{-1}{2\pi} \int_0^{2\pi} dt \int_0^t (\exp sA) \cdot Q \, ds.$$

Notice that, since $A \in \mathscr{g}_\iota$, ad_A, and hence $\exp A$, preserves the gradation of $\mathscr{g}$.

6.1. Lemma. $[A, Q^{\text{av}}] = 0$ *and* $[A, \tau(Q)] = Q - Q^{\text{av}}$.

Proof. We use the identity

$$[A, (\exp tA) \cdot Q] = \frac{d}{dt} (\exp tA) \cdot Q.$$

Everything follows from this; for example,

$$\begin{aligned} [A, \tau(Q)] &= -\frac{1}{2\pi} \int_0^{2\pi} ((\exp tA) \cdot Q - Q) \, dt \\ &= Q - Q^{\text{av}}. \end{aligned}$$

Q.E.D.

Now let $Q \in \mathscr{g}_{k_0}$, where $k_0 < \iota$, and consider the perturbation of A,

$$P = A + Q.$$

6.2. Lemma. *Under the action of the adjoint group, P can be mapped into an element of* $\mathscr{g}$ *of the form* $A + Q^{\text{av}} + Q^{\#}$, *where* $Q^{\#}$ *is of order* $2k_0 - \iota < k_0$. *Furthermore,*

$$Q^{\#} = \tfrac{1}{2}[\tau(Q), Q + Q^{\text{av}}]$$

modulo elements of order $3k_0 - 2\iota$.

Proof. Let $F = \exp \tau(Q)$. We have

$$F \cdot A = A + Q^{\mathrm{av}} - Q + \sum_{n=2}^{\infty} \frac{1}{n!} \operatorname{ad}^n_{\tau(Q)}(A),$$

by Lemma 6.1. Hence,

$$F \cdot P = A + Q^{\mathrm{av}} + \sum_{n=1}^{\infty} Q_n,$$

where

$$Q_n = \frac{1}{(n+1)!} \operatorname{ad}^{n+1}_{\tau(Q)}(A) + \frac{1}{n!} \operatorname{ad}^n_{\tau(Q)}(Q).$$

Notice that both summands in the definition of Q_n are of order $(n+1)\, k_0 - n\iota$, which is a decreasing function of n since $k_0 < \iota$. Using again Lemma 6.1, we can rewrite Q_n as

$$Q_n = \frac{1}{(n+1)!} \operatorname{ad}^n_{\tau(Q)}(Q^{\mathrm{av}}) + \frac{n}{n+1} \operatorname{ad}^n_{\tau(Q)}(Q).$$

Lemma 6.2 follows. Q.E.D.

Iterating the averaging method, we can prove the following lemma, which is precisely what we need for Lemmas 2.3 and 4.4.

6.3. Lemma. *Consider $P = A + Q + R$, where Q is as above and R is of order $2k_0 - \iota$. Under the action of the adjoint group, P can be mapped to an element of $\mathfrak{g}$ which equals*

$$A + Q^{\mathrm{av}} + \tfrac{1}{2}[\tau(Q), Q]^{\mathrm{av}} + R^{\mathrm{av}} \tag{6.1}$$

modulo elements of order $(2k_0 - \iota - 1)$. Furthermore, if $Q^{\mathrm{av}} = 0$, (6.1) can be replaced by

$$A + R^{\mathrm{av}} - \frac{1}{2\pi} \int_0^{2\pi} dt \int_0^t [U(t) \cdot Q, U(s) \cdot Q]\, ds, \tag{6.2}$$

where $U(t) = \exp tA$.

Proof. Let $Q^{\#}$ be the operator of Lemma 6.2, so that P can be conjugated to $P_1 = A + Q^{\mathrm{av}} + Q^{\#} + R$ modulo elements of order $(2k_0 - \iota - 1)$. If we apply $\exp(\tau(Q^{\#} + R))$ to P_1, we obtain an element which equals

$$A + Q^{\mathrm{av}} + \tfrac{1}{2}[\tau(Q), Q]^{\mathrm{av}} + \tfrac{1}{2}[\tau(Q), Q^{\mathrm{av}}]^{\mathrm{av}} + R^{\mathrm{av}}$$

mod $(2k_0 - \imath - 1)$. Using the identity

$$U(t) \cdot \tau(Q) = \tau(Q) - tQ^{\text{av}} + \int_0^t U(s) \cdot Q \, ds, \tag{6.3}$$

one can prove that $(\tau(Q))^{\text{av}} = -\pi Q^{\text{av}}$ and thus $[\tau(Q), Q^{\text{av}}]^{\text{av}} = 0$. Equation (6.2) also follows from (6.3) if we assume $Q^{\text{av}} = 0$. Q.E.D.

To apply Lemma 6.3 to "classical averaging" (Lemma 2.3), we take $\mathfrak{g}$ to be the vector space of functions of the form fH, where $H \in C^\infty(X)$ and f is a smooth function on the real line defined in a neighborhood of zero. For every $k \leqslant 0$, $\mathfrak{g}_k$ consists of those functions $fH \in \mathfrak{g}$ such that f vanishes to order $-(k+1)$ at zero, and the bracket is the Poisson bracket on X,

$$[f_1 H_1, f_2 H_2] = f_1 f_2 \{H_1, H_2\}.$$

Then $\mathfrak{g}$ is 0-graded, and the function H_0 of Section 2 can play the role of the element A of this section. Thus, Lemma 2.3 follows from Lemma 6.3.

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